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Remarks on Tannaka recovery of coalgebras

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Abstract

In virtue of some highly generalized Tannaka–Krein-type reconstruction theorems, it is natural to ask which functors $F:\mathscr{C}\to\mathscr{V}$ are equivalent to the forgetful functor $U_H:\mathrm{Comod}(H)\to\mathscr{V}$ for some $\mathscr{V}$-Hopf-algebra H. We will study in this paper the question whether $\mathrm{Comod}(\mathrm{coend}(F))\cong C$ when $F=U_C$ for a coalgebra C and $\mathscr{V}$ is a module category over a ring. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Some highly generalized (very elegant and useful) Tannaka–Krein-type reconstruction theorems have been presented in recent years by Street and Majid, etc. (e.g., see [3, 5–9, 11, 12]), based on the papers of Ulbrich [15] and of Deligne and Milne [4]. The main result is the following (see [12]):

Theorem 1.0. *Suppose that $\mathscr{C}$ is a small category and $\mathscr{V}$ is a tensor category cocomplete over $\mathscr{C}$ and such that tensor preserves colimits in each variable. Let $F:\mathscr{C}\to\mathscr{V}$ be a functor such that for all $X\in\mathscr{C}$, the object FX has a left dual $(FX)^*$. Then the object*

$$\mathrm{coend}(F)=\int^X F(X)^*\otimes(FX)$$

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has a natural structure of $\mathscr{V}$-coalgebra and there is a commutative triangle

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "N"] \arrow[dr, "F"'] & & Comod_{\mathscr{V}}(coend(F)) \arrow[dl, "U"] \\ & \mathscr{V} & \end{tikzcd}$$

where U is the forgetful functor. Indeed, coend(F) *is the universal $\mathscr{V}$-coalgebra such that F factors through the forgetful functor U. If $\mathscr{C}$ is a tensor category, $\mathscr{V}$ is braided and F is a tensor functor then* coend(F) *becomes a $\mathscr{V}$-bialgebra and N becomes a tensor functor. If $\mathscr{C}$ is left autonomous then* coend(F) *becomes a $\mathscr{V}$-Hopf-algebra. If $\mathscr{C}$ is autonomous then* coend(F) *becomes a $\mathscr{V}$-Hopf-algebra with invertible antipode. If $\mathscr{C}$ is a tortile tensor category then* coend(F) *becomes a cotortile $\mathscr{V}$-Hopf-algebra* (*quantum group*) *and N becomes a balanced tensor functor.*

An important (and more classical) case of Tannaka–Krein duality is the characterization of those $F:\mathscr{C}\to\mathscr{V}$ equivalent to the forgetful functor U_H : Comod$_{\mathscr{V}}(H)_c\to\mathscr{V}$ for some Hopf algebra H, where Comod$_{\mathscr{V}}(H)_c$ is the full subcategory consisting of those H-comodules M whose underlying object has a dual. The question arises here whether coend$(F)\cong C$ when $F=U_C$ for a coalgebra. Deligne and Milne have investigated the case when $\mathscr{V}=$ Vect (the category of all vector spaces). It is also known that if C is a coalgebra over a field K and $U:$ Comod$(C)_c\to$ Vect is the forgetful functor then there is a coalgebra isomorphism coend$(U)\cong C$ (see also [12, 11]). In this paper, we will investigate the case when $\mathscr{V}$ is a module category over a not-necessarily commutative ring. To do this, one of the key points is to have some generalized version of the fundamental theorem of coalgebras [13, p. 46]. Towards this, the technique used by Barr presented in his early paper [2] is useful.

2. Some ring-theoretic preliminaries

In this paper, a ring means a not-necessarily commutative ring with identity. A ring R is said to be a *domain* if $a,b\in R$, $ab=0$ implies $a=0$ or $b=0$. Let R be a ring. A left (resp. right) R-module is called *left* (resp. *right*) *cauchy* when it is finitely generated and projective. R is said to be *left* (resp. *right*) *Dedekind* if each left (resp. right) ideal of R is a left (resp. right) cauchy. R is said to be *Dedekind* if it is both left and right Dedekind. An R-module is said to be *cauchy* if it is both left and right cauchy.

Commutative Dedekind rings historically arose in number theory and are well known. For example, each integral closure of Z – the integer domain – in a separable extension of Q (the rational field) with finite degree, is Dedekind. It is well known that each commutative principal ideal domain, for example the polynomial ring k[x] over a field, is Dedekind. A well-known example of a Dedekind domain but not a principal ideal domain is the domain $\{a+b\sqrt{-5}\colon\ a,b\in Z\}$. However, we will mainly discuss here their non-commutative analogues. Note that in the literature there are some slightly different notions of non-commutative Dedekind rings.

Recall the following two known facts:

(a) If R is left (resp. right) Noetherian, then each finitely generated left (resp. right) R-module is left (resp. right) Noetherian (i.e., every left (resp. right) submodule of it is finitely generated);

(b) If each left (resp. right) ideal of R is projective, then each left (resp. right) submodule of a projective left (resp. right) R-module is left (resp. right) projective (e.g., see [1, Theorem 1.1, p. 352]).

We have the following:

Proposition 2.1. *A ring R is Dedekind iff each submodule of a left cauchy module is left cauchy and each submodule of a right cauchy module is right cauchy.*

Definition 2.2. A (not-necessarily commutative) domain R is called a *principal left* (*resp. right*) *ideal domain* if each left (resp. right) ideal is principal; and is called a *principal ideal domain* if it is both a left and right principal ideal domain.

Clearly, each principal ideal domain is Dedekind.

The real differential operator ring $B_1(R)=R(x)[D]$ – the skew polynomial ring over the real-value rational function field $R(x)$ (that is, $B_1(R)$ is a free left $R(x)$-module with generators $\{D,D^2,\ldots,D^n,\cdot\}$ satisfying $DP=PD+P'$ for each $P\in R(x)$ where P' is the derivative of P) is a typical example of a non-commutative principal ideal domain. More generally, each polynomial ring $D[x]$ over a division ring D is a PID.

Recall that each left module M over a commutative ring R can always be regarded as an (R,R)-bimodule in the way that $m\cdot r=rm$ for each $m\in M$ and $r\in R$. The procedure can be extended to some special kind of left modules over a non-commutative ring:

Definition 2.3. Let R be a ring and let F be a free left R-module. We define $(\sum^n s_i x_i)\cdot r$ to be $\sum^n s_i r x_i$ for each $r\in R$ and each element $\sum^n s_i x_i\in F$ where x_i's are basis elements. It makes sense since F is a free left R-module. Then it is easy to check that this definition gives F a free right R-module structure. Thus a free left module is in fact a both free right and free left (R,R)-bimodule, which we will call *compatibly free*. A compatibly free R-module is not a free (R,R)-bimodule in general. For example, the free (R,R)-bimodule generated by one generators is $R\otimes_Z R$.

Definition 2.4. An (R,R)-bimodule is called *compatible* if it is isomorphic to a sub-bimodule of a compatibly free module.

Notice that each left submodule M of a compatibly free module F has a compatible bimodule closure MR which is the intersection of all sub-bimodules of F containing M.

Lemma 2.5. (1) *If R is Dedekind and M is a left cauchy R-module, then MR is cauchy* (i.e., *both left and right cauchy*).

(2) *If M is a compatible module* (i.e., *a sub-bimodule of a compatibly free module*), *and if M is finitely generated left* (*or right*) *R-module, then M is cauchy.*

Proof. In (1), the given module M is isomorphic a left submodule of a compatibly free module F with a finite basis which is cauchy. Hence, MR is a sub-bimodule of F and is cauchy by Proposition 2.1.

In (2), the bimodule M is in fact a sub-bimodule of a compatibly free module F with a finite basis, and hence is cauchy. □

Lemma 2.5 will be used several time in this paper.

3. Algebra and coalgebra over general rings

In this section we will briefly discuss algebras and coalgebras over a general ring. Such knowledge is needed for proving the main results in this paper.

Let R be a given ring.

For a left R-module M, the hom set $\mathrm{Hom}_R(M,R)$, denoted by M^*, consisting of all left R-linear morphisms, will be called *the right dual* of M. M^* is a right R-module defined by $(fr)(m)=f(m)r$ for all $r\in R$ and $m\in M$.

For a right R-module M, the hom set $\mathrm{Hom}^R(M,R)$, denoted by $M^\vee$, consisting of all right R-linear morphisms will be called *the left dual* of M. $M^\vee$ is a left R-module defined by $(rf)(m)=rf(m)$ for all $r\in R$ and $m\in M$.

For an (R,R)-bimodule M, the left dual M^*, where M is regarded as a left R-module, is an (R,R)-bimodule defined by $(sfr)(m)=f(ms)r$ for all $r,s\in R$ and $m\in M$. While the right dual $M^\vee$, where M is regarded as a right R-module, is an (R,R)-bimodule defined by $(sfr)(m)=sf(rm)$ for all $r,s\in R$ and $m\in M$.

An algebra over R (or R-algebra) is an (R,R)-bimodule A together with bimodule morphisms $\mu:A\otimes A\to A$ and $\eta:R\to A$ satisfying the usual axioms.

A coalgebra over R (or R-coalgebra) is an (R,R)-module C together with bimodule morphisms $\delta:C\to C\otimes C$ and $\varepsilon:C\to R$ satisfying the usual axioms as follows:

$$C \xrightarrow{\delta} C\otimes C \underset{1\otimes\delta}{\overset{\delta\otimes 1}{\rightrightarrows}} C\otimes C\otimes C$$

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, equal] & C\otimes_R C \arrow[d, shift right, "\varepsilon\otimes 1"'] \arrow[d, shift left, "1\otimes\varepsilon"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

The algebra-morphisms and coalgebra-morphisms are R-bimodule morphisms which respect the algebra-structure and the coalgebra-structure, respectively, as usual.

Given an R-coalgebra C and an R-algebra A, for each pair $f,g\in \hom_R(C,A)$, the classical definition of $f\otimes g: C\otimes C\to A\otimes A$ does not make sense since f and g are merely left R-linear but not R-bilinear in general.

However, we will show here that there still exists a so-called *convolution product* on C^* and on $C^\vee$ even if R is not commutative. The new convolution product will yield the classical twist convolution product when R is commutative.

We will first consider a slightly general case. Let A be an R-algebra and C,D two right A-modules compatible with the right R-module action (i.e. $(ca)r=c(ar)$ for each $c\in C, a\in A$ and $r\in R$). Then $\hom_R(C,A)$ and $\hom_R(D,A)$ become left A-modules defined by $(af)(c)=f(ca)$ for each $a\in A$, $f\in \hom_R(C,A)$ (resp. in $\hom_R(D,A)$) and $c\in C$. Note that we have $(ar)f=a(rf)$ and $(af)r=a(fr)$.

First we have an (R,R)-bimodule morphism (see also [12])

$$\rho: \hom_R(C,A)\otimes \hom_R(D,A)\to \hom_R(C,\hom_R(D,A))$$

defined by $\rho(f\otimes g)=f(-)g$. It is well defined since $f(-)g$ sending each $n\in C$ to $f(n)g$ is in $\hom_R(D,A)$ and since $(fr)(-)g=f(-)(rg)$ for each $r\in R$ (since $((fr)(n)g)$ $=((f(n)r)g=f(n)(rg)=f(-)(rg)(n))$. We verify ρ is R-bilinear: for each $r\in R$, each $n\in C$, $\rho(rf\otimes g)(n)=(rf)(n)g=f(nr)g=(f(-)g)(nr)=(r(f(-)g))(n)=(r\rho(f\otimes g))$ (n) and so $\rho(rf\otimes g)=r\rho(f\otimes g)$, that is, ρ is left-linear. On the other hand, $\rho(f\otimes gr)$ $(n)=f(n)(gr)=(f(n)g)r=((f(-)g)(n))r=((f(-)g)r)(n)$ and so $\rho(f\otimes gr)=(f(-)$ $g)r=\rho(f\otimes g)r$, that is, ρ is right-linear.

Secondly, we have a natural (R-bilinear) isomorphism (see also [12])

$$\xi: \hom_R(C,\hom_R(D,A))\to \hom_R((D\otimes C),A)$$

defined by setting $\xi(u)(d\otimes c)=(u(c))(d)$. First we verify that $\xi(u)$ is in $\hom_R((C\otimes C),A)$: for each $u\in \hom_R(C,\hom_R(D,A))$, $\xi(u)$ is well defined since $(u(c))(dr)=$ $((r(u(c)))(d)=u(rc)(d)$. Now we show that $\xi(u)$ is left linear: for each $r\in R$ and $d\otimes c\in D\otimes C$, $\xi(u)(rd\otimes c)=u(c)(rd)=r(u(c)(d))=r\xi(u)(d\otimes c)$. In fact, $\xi(u)$ is the composition

$$\xi(u) =\!=\!= ((D\otimes C)\xrightarrow{1\otimes u} D\otimes \hom_R(D,A)\xrightarrow{ev_D} A).$$

Note that both evaluation morphisms $ev_C: C\otimes \hom_R(C,A)\to A$ and $ev^C: \hom^R(C,A)\otimes C\to A$ are R-bilinear.

Now we show that ξ is R-bilinear: for each $r\in R$, $d\otimes c\in D\otimes C$, $\xi(ru)(d\otimes c)=((ru)$ $(c))(d)=(u(cr))(d)=\xi(u)(d\otimes(cr))=\xi(u)((d\otimes c)r)=(r\xi(u))(d\otimes c)$ and so $\xi(ru)=$ $r\xi(u)$, that is, ξ is left-linear. While $\xi(ur)(d\otimes c)=((ur)(c))(d)=(u(c)r)(d)=(u(c)$ $(d))r=(\xi(u)(d\otimes c))r))=(\xi(u)r)((d\otimes c)$ and so $\xi(ur)=\xi(u)r$, that is, ξ is right-linear.

Now we have an R-bilinear morphism

$$\theta: \hom_R(C,A)\otimes \hom_R(D,A)\to \hom_R(D\otimes C,A)$$

as the composition of ρ and ξ. Note that for each $d \otimes c \in D \otimes C$,

$$\theta(f \otimes g)(d \otimes c) = (f(c)g)(d) = g(df(c)).$$

Remark 3.1. (1) Summarizing the above construction, for each pair of right A-modules C, D where A is an R-algebra, we have the following R-bimodule morphism:

$$\theta_A : \hom_R(C, A) \otimes \hom_R(D, A) \to \hom_R(D \otimes C, A)$$

defined by, for each $d \otimes c \in D \otimes C$, $\theta(f \otimes g)(d \otimes c) = (f(c)g)(d) = g(df(c))$.

When $A = R$, we have

$$\theta : C^* \otimes D^* \to (D \otimes C)^*$$

defined by for each $d \otimes c \in D \otimes C$, $\theta(f \otimes g)(d \otimes c) = g(df(c))$.

We can repeat the above procedure to get the following morphism, still denoted by θ:

$$B^* \otimes C^* \otimes D^* \to B^* \otimes (D \otimes C)^* \to (D \otimes C \otimes B)^*,$$

that is, $\theta(h \otimes f \otimes g)(d \otimes c \otimes b) = g(df(c(h(b))))$. We will use it when we prove Proposition 3.2.

To prove Lemmas 3.4 and 3.5, we need the following R-bilinear morphism for arbitrary $C, D, M \in (R, R)$-BiMod,

$$\theta_M : C^* \otimes D^* \to \hom_R(M \otimes D \otimes C, M)$$

defined by $\theta_M(f \otimes g)(m \otimes d \otimes c) = mg(df(c))$. It is well defined since for all $f, g \in C^* \otimes D^*$, $\theta_M(f \otimes g)$ is a multi-linear morphism from $M \times D \times C$ to M and since $\theta_M(f \otimes g)$ is a left R-module morphism from $M \otimes D \otimes C$ to M. Further we can easily check that θ_M is R-bilinear.

Similarly we have a right dual ($C^\vee$)-version. There is an R-bilinear morphism

$$\theta^M : C^\vee \otimes D^\vee \to \hom^R(D \otimes C \otimes M, M)$$

defined by $\theta^M(f \otimes g)(d \otimes c \otimes m) = (fg(d))(c)m = f(g(d)c)m$ (we need it when we prove Proposition 3.5). When $M = R$, we have

$$\theta^R : C^\vee \otimes D^\vee \to (D \otimes C)^\vee$$

defined by $\theta^R(f \otimes g)(d \otimes c) = (fg(d))(c) = f(g(d)c)$.

(2) If g happens to be R-bilinear, then we have

$$\theta_R(f \otimes g)(d \otimes c) = g(df(c)) = g(d)f(c).$$

Dually, if f is R-bilinear, then $\theta^R(f \otimes g)(d \otimes c) = (fg(d))(c) = g(d)f(c)$.

(3) If R is commutative, we have that the classical definition of the twist tensor product $(f \otimes g)(m \otimes n) = f(n)g(m)$ is equal to $\theta_R(f \otimes g)(m \otimes n)$.

Now for each coalgebra C over a general ring R, we have the *convolution product* $f * g$ of $f, g \in C^*$ which is defined by $(f * g) = \delta^* \theta_R(f \otimes g)$, that is, $(f * g)(c) = \sum g(c_1 f(c_2))$ where $\delta(c) = \sum c_1 \otimes c_2$; and the *convolution product* $f * g$ of $f, g \in C^\vee$ which is defined by $(f * g) = \delta^\vee \theta^R(f \otimes g)$, that is, $(f * g)(c) = \sum f(g(c_1)c_2)$.

Using the new convolution product and the morphisms θ and θ^R introduced above, we have the following propositions (the proofs are omitted):

Proposition 3.2. *If (C, δ, ε) is an R-coalgebra, then C^* and $C^\vee$ are R-algebras, where the multiplications π and π' are the respective compositions*

$$\pi = (C^* \otimes C^* \xrightarrow{\theta} (C \otimes C)^* \xrightarrow{\delta^*} C^*),$$

$$\pi' = (C^\vee \otimes C^\vee \xrightarrow{\theta'} (C \otimes C)^\vee \xrightarrow{\delta^\vee} C^\vee),$$

where $\delta^*(f) = f\delta \in M^*$ and $\delta^\vee(f) = f\delta \in M^\vee$ and $\eta : R \to C^*$ and $\eta' : R \to C^\vee$ are defined by $\eta(1) = \varepsilon$ and $\eta'(1) = \varepsilon$,

Lemma 3.3. *If $f : C \to D$ is an R-coalgebra morphism, then both $f^* : D^* \to C^*$ and $f^\vee : D^\vee \to C^\vee$ are R-algebra-morphisms.*

Given an R-coalgebra C, a C-comodule M is an (R, R)-bimodule M together with an R-bimodule morphism $\delta_M : M \to M \otimes C$ such that

$$\begin{tikzcd} M \arrow[r, "\delta_M"] & M \otimes C \arrow[r, shift left, "\delta_M \otimes 1"] \arrow[r, shift right, "1 \otimes \delta_C"'] & M \otimes C \otimes C \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "\delta_M"] & M \otimes C \arrow[r, shift left, "\delta_M \otimes 1"] \arrow[r, shift right, "1 \otimes \delta_C"'] & M \otimes C \otimes C \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[rr, "\delta"] \arrow[dr, "\cong"'] & & M \otimes_R C \arrow[dl, "1 \otimes \varepsilon"] \\ & M \otimes R & \end{tikzcd}$$

Lemma 3.4. *Each right C-comodule M has a right C^*-module action and each left C-comodule has a left $C^\vee$-module action.*

Proof. We will only show that each right C-comodule M has a right C^*-module action μ:

$$M \otimes C^* \xrightarrow{\delta \otimes 1} M \otimes C \otimes C^* \xrightarrow{1 \otimes \langle , \rangle} M \otimes R \cong M.$$

That is, define $\mu(m \otimes f) = m \cdot f = \sum_{(m)} m_i f(c_i)$ where $\delta(m) = \sum m_i \otimes c_i$. We will check that this gives M a right C^*-module action. The unit axiom: $m \cdot 1_{C^*} = m \cdot \varepsilon = \sum_{(m)} m_i(\varepsilon(c_i)) = m$ by the counit axiom of the comodule.

Now for $f, g \in C^*$ and $m \in M$, $m \cdot (f * g) = \sum_{(m)} m_i(f * g)(c_i) = \sum_{(m)}(m_i)(\theta(f \otimes g))(\delta(c_i)) = \sum_{(m)} m_i \sum_{(c_i)} g(c_{(1i)} f(c_{(2i)})) = \theta_M(f \otimes g)(1 \otimes \delta_C)\delta_M(m)$, where $\delta(c_i) = \sum c_{(1i)} \otimes c_{(2i)}$. While $(m \cdot f) \cdot g = \sum_{(m)}(m_i f(c_i)) \cdot g$, since $\delta(m_i f(c_i)) = \delta(m_i) f(c_i) = \sum m_{(1i)} \otimes c_{(2i)} f(c_i)$ where $\delta(m_i) = \sum m_{(1i)} \otimes c_{(2i)}$, we see that $(\sum_{(m)} m_i f(c_i)) \cdot g = \sum_{(m)} m_{(1i)} g(c_{(2i)} f(c_i)) = \theta_M(f \otimes g)(\sum m_{1i} \otimes c_{2i} \otimes c_i) = \theta_M(f \otimes g)(\delta_M \otimes 1_C)\delta_M(m)$. By the associative axiom $(\delta_M \otimes 1)\delta_M = (1 \otimes \delta_C)\delta_M$, we see that $m \cdot (f * g) = (m \cdot f) \cdot g$. Diagrammatically,

$$
\begin{tikzcd}
C \otimes C^* \otimes C^* \arrow[d, "\delta \otimes 1 \otimes 1"'] & & \\
C \otimes C \otimes C^* \otimes C^* \arrow[r, "\sigma_{1342}"] \arrow[d, shift right=2, "1 \otimes \delta \otimes 1 \otimes 1"'] \arrow[d, shift left=2, "\delta \otimes 1 \otimes 1 \otimes 1"] & C \otimes C^* \otimes C \otimes C^* \arrow[d, "\delta \otimes 1 \otimes 1 \otimes 1"] \\
C \otimes C \otimes C \otimes C^* \otimes C^* \arrow[r, "\sigma_{12453}"] \arrow[d, "\sigma_{12453}"'] & C \otimes C \otimes C^* \otimes C \otimes C^* \arrow[d, "1 \otimes \langle , \rangle \langle , \rangle"] \\
C \otimes C \otimes C^* \otimes C \otimes C^* \arrow[r, "1 \otimes \langle , \rangle \langle , \rangle"'] & C \otimes R \otimes R.
\end{tikzcd}
$$

□

We will use Lemma 3.4 when we prove Proposition 4.11.

Let C be an R-coalgebra and $\delta : M \to M \otimes C$ be any R-bimodule morphism. Then we have a morphism $\mu : M \otimes C^*$ as described in the above Lemma; that is $\mu(m \otimes f) = \sum m_i f(c_i)$ where $\delta(m) = \sum m_i \otimes c_i$. Then we have the following:

Proposition 3.5. *If C is left cauchy, then μ is a right C^*-module action iff δ is a right C-comodule coaction.*

Proof. If δ is a right coaction, then μ is a right module action by the above lemma. Now assume that μ is a C^*-module action. Since C is left cauchy, then there is an R-bilinear morphism $d : R \to C^* \otimes C$, say $d(1) = \sum u_i \otimes c_i$, such that $\sum u_i(c) c_i = c$ for each $c \in C$ (see [12] or [14]).

It is easy to check that δ is the composition

$$M \cong M \otimes R \xrightarrow{1 \otimes d} M \otimes C^* \otimes C \xrightarrow{\mu \otimes 1} M \otimes C.$$

In fact, the above composite is equal to the following composite:

$$M \xrightarrow{1 \otimes d} M \otimes C^* \otimes C \xrightarrow{\delta \otimes 1 \otimes 1} M \otimes C \otimes C^* \otimes C \xrightarrow{1 \otimes ev_C \otimes 1} M \otimes C$$

which sends $m \in M$ to $m \otimes \sum u_i \otimes c_i$, and then to $\sum m_j \otimes d_j \otimes \sum u_i \otimes c_i$, and then to $\sum m_j \otimes \sum u_i(d_j)c_i = \sum m_j \otimes d_j = \delta(m)$ where $\delta(m) = \sum m_j \otimes d_j$ and $d(1) = \sum u_i \otimes c_i$.

It now remains to show that the composition is a C-comodule coaction. Note that $\delta(m) = \mu(m \otimes \sum u_i) \otimes c_i = (m \cdot \sum u_i) \otimes c_i$ where $\sum u_i \otimes c_i = d(1)$.

The counit axiom: for each $m \in M$, $(1 \otimes \varepsilon_C)\delta(m) = (m \cdot \sum u_i) \otimes \varepsilon(c_i) \cong (m \cdot \sum u_i)\varepsilon(c_i)$ $= m \cdot (\sum u_i)\varepsilon(c_i) = \sum m_j(\sum u_i \varepsilon(c_i))c_j)$, where $\delta(m) = \sum_j m_j \otimes c_j$. Thus we further have $(1 \otimes \varepsilon_C)\delta(m) = \sum m_j(\sum u_i \varepsilon(c_i))c_j) = \sum m_j(\sum u_i(c_j)\varepsilon(c_i)) = \sum m_j \varepsilon(\sum u_i(c_j)(c_i))$ $= \sum m_j \varepsilon(c_j) = m \cdot \varepsilon = m \cdot 1_{C^*} = m$.

To show the associativity, note that for any $f, g \in C^*$, $(m \cdot f) \cdot g) = (\sum_{(m)} m_i f(c_i)) \cdot g$ $= \sum_{(m)} \sum_{(m_i)} m_{1i} g(m_{2i} f(c_i)) = \theta_M(f \otimes g)(\sum_{(m)} \sum_{(m_i)} m_{1i} \otimes (m_{2i} \otimes (c_i)) = \theta_M(f \otimes g)$ $(\delta_M \otimes 1)\delta_M(m)$, where $\theta_M : C^* \otimes C^* \to \hom_R(M \otimes C \otimes C, M)$ defined by $\theta_M(f \otimes g)$ $(m \otimes c \otimes d) = mg(cf(d))$ described above.

While $m \cdot (f * g) = (\sum_{(m)} m_i(f * g)(c_i)) = \sum_{(m)} \sum_{(c_i)} m_i g(c_{1i} f(c_{2i})) = \theta_M(f \otimes g)$ $(\sum_{(m)} \sum_{(c_i)} m_i \otimes (c_{1i} \otimes (c_{2i})) = \theta_M(f \otimes g)(1 \otimes \delta_C)\delta_M(m)$

The two left-hand sides are equal by the associativity of the C^*-module action μ. This means that

$$\theta_M(f \otimes g)(\delta_M \otimes 1)\delta_M(m) = \theta_M(f \otimes g)(1 \otimes \delta_C)\delta_M(m)$$

for all $f, g \in C^*$.

We claim that $(\delta_M \otimes 1)\delta_M(m) = (1 \otimes \delta_C)\delta_M(m)$. If not, say $q = \sum m_i \otimes x_i \otimes y_i =$ $(\delta_M \otimes 1)\delta_M(m) - (1 \otimes \delta_C)\delta_M(m) \neq 0$.

But since C is left cauchy, we see that $C \cong C^{*\vee}$ and $M \otimes C \otimes C \cong M \otimes C^{*\vee} \otimes C^{*\vee} \cong$ $M \otimes (C^* \otimes C^*)^\vee \cong \mathrm{Hom}^R(C^* \otimes C^*, M)$ (for the details see [14]). Note that these isomorphisms send $m \otimes x \otimes y$ to a morphism $m \cdot \theta^R(\bar{x} \otimes \bar{y})(-) : C^* \otimes C^* \to M$ (where $\bar{x}(f) = f(x)$). Thus we have a non-zero morphism $\sum m_i \cdot \theta^R(\bar{x}_i \otimes \bar{y}_i)(-) : C^* \otimes C^* \to M$ and hence there is $f \otimes g \in C^* \otimes C^*$ such that $\sum m_i \cdot \theta^R(\bar{x}_i \otimes \bar{y}_i)(f \otimes g) \neq 0$ but $\sum m_i \cdot \bar{x}_i$ $\otimes \bar{y}_i(f \otimes g) = \sum m_i \bar{x}_i(\bar{y}_i(f)g) = \sum m_i \bar{x}_i(f(y_i)g) = \sum m_i(f(y_i)g)(x_i) = \sum m_i g(x_i f(y_i))$ $= \theta_M(f \otimes g)(q)$ contradicting the assumption on q. Thus we have shown the co-associativity. $\square$

Lemma 3.6. *If an R-coalgebra C is left cauchy, then the category of all right C-comodules is isomorphic to the category of all right C^*-modules. Dually, if C is* *right cauchy then the category of all left C-comodules is isomorphic to the category of all left C^*-modules.*

4. Fundamental theorem of comodules

The classical fundamental theorem on coalgebras over a field (see [13, p. 46]) says that for each element c of a coalgebra C, the smallest subcoalgebra containing c is finite dimensional.

To recover a given coalgebra via Tannaka–Krein-type reconstruction theorem, one of the key points is to have some generalized version of the fundamental theorem for comodules as some authors noticed (e.g. [3]).

The main aim of this section is to contribute several such generalizations.

First recall that each R-coalgebra C becomes a left $C^\vee$- and a right C^*-module as described in Lemma 3.4.

Lemma 4.1. *For any ring R and any R-coalgebra (C, δ, ε) and any $c \in C$,*

(1) *$c \cdot C^* \subseteq \sum c_{1i}R$ and $C^\vee \cdot c \subseteq \sum Rc_{2i}$ for any representative $\delta(c) = \sum c_{1i} \otimes c_{2i}$.*

(2) *$c \cdot C^*$ is a right sub-comodule of C iff $c \cdot C^* \supseteq \sum c_{1i}R$ for some representative $\delta(c) = \sum c_{1i} \otimes c_{2i}$.*

Proof. (1) is trivial. To show (2), the point is to show $\delta(c \cdot f) = \delta(c)f$, the details refer to the proof of Lemma 4.2 below. □

Remark. If the underlying module of C is free, then condition (2) is satisfied. Thus $(C^\vee \cdot cR)$ (resp. $(Rc \cdot C^*)$) is a left (resp. right) subcomodule of C containing c (see Lemma 4.2 below).

To weaken the assumption of freeness of C, we introduce temporarily a notion of *locally free* which means that each finitely generated submodule is compatibly free. (The reader should not be confused with the standard notion of locally free in commutative algebra, which means that each localization at a prime ideal R is R_P-free.)

By the results that any finitely generated torsion free module over a commutative principal ideal domain is free, and that each finitely generated projective module over a commutative semilocal ring (those rings which have finitely many maximal ideals) (e.g. see [1, p. 447]) is free, our basic examples include all torsion free modules over a commutative PID and all projective modules over a commutative semilocal ring. The field of quotient of a PID R which is not a field, gives an example which is R-locally free but is not R-free.

The following construction uses the local freeness. The technique used here was first developed by Barr [2]. Since we are working on torsion free modules over a principal ideal domain, by the result that the torsion freeness is equivalent to the flatness in this case, we do not need worry about the pureness as Barr considered in [2]. Still write $D \cdot D$ for the image $D \otimes D \to C \otimes C$ for each submodule D of C.

Construction. Let (C, δ, ε) be a coalgebra over a PID where C is locally free, and $c \in C$ with $\delta(c) = \sum_i^n c_{1i} \otimes c_{2i}$.

Write W_0 for the sum of the module $\sum_i^n Rc_{1i}R$ and the module $\sum_i^n Rc_{2i}R$. For each c_{1i} and each c_{2i} we choose representatives of $\delta(c_{1i}) = \sum c_{11i} \otimes c_{12i}$ and $\delta(c_{2i}) = \sum c_{21i} \otimes c_{22i}$. Write M_1 for the smallest submodule of C which contains all $c_{11i}, c_{12i}, c_{21i}, c_{22i}$, and hence $\delta(M_0) \subseteq M_1 \cdot M_1$. Repeat the procedure, we write M_2 for the module induced from M_1 in the above way, such that $\delta(M_1) \subseteq M_2 \cdot M_2$. Under our assumption we have $M_i \otimes C \cong M_i \cdot C$.

There is a left R-linear morphism

$$\zeta : M_2^\vee \to \hom^R(M_2 \otimes C, C)$$

defined by $\zeta(f)(m \otimes c) = f(m)c$.

First it is easy to see that for each $f \in M_2^\vee$, $\zeta(f)$ is defined to be a bilinear morphism from $M_2 \times C$ to C since f is a right R-linear, and hence $\zeta(f)$ factorizes through $M_2 \otimes C$. Moreover, $\zeta(f)$ is obviously right R-linear. Now we check that ζ is a left R-linear: in fact, $\zeta(rf)(m \otimes c) = (rf)(m)c = r(f(m)c) = r\zeta(f)(m \otimes c)$.

Then the coaction $\delta : M_1 \to M_2 \otimes C$ induces an R-bilinear morphism $(-)\delta : \hom^R(M_1, C) \to \hom^R(M_2 \otimes C, C)$.

Now write $f \cdot w$ for the image $\zeta(f)\delta(w)$ for each $f \in M_2^\vee$ and $w \in M_1$, and then write $(M_2^\vee \cdot w)$ for the left submodule $\{f \cdot w\colon f \in M_2^\vee\} = \{\sum f(w_1)w_2\colon f \in M_2^\vee\}$ of C. This is a left submodule of C since we have $r(f \cdot c) = (rf) \cdot c$ for each $r \in R$ and $f \in M_2^\vee$. Note that we also have $(f \cdot w)r = f \cdot (wr)$ for each $r \in R$. As a consequence, we have the sub-bimodule $(M_2^\vee \cdot cR)$ where c is the given element. In particular, $(C^\vee \cdot cR)$ is a sub-bimodule of C containing c.

Lemma 4.2. *For any ring R, if an R-coalgebra (C, δ, ε) has underlying module free or locally free, then, for each $c \in C$, $(M_2^\vee \cdot cR)$ (resp. $(Rc \cdot M_2^*)$ is a left (resp. right) subcomodule of C which contains the given c. When C is free, $M_2^\vee$ (resp. M_2^*) can be replaced by $C^\vee$ (resp. C^*).*

Proof. Use the notations introduced above. Then M_1 is compatibly free (as well as M_0 and M_2) by the local freeness. Suppose that $\{e_i;\ i \leq n\}$ is a basis for M_1. Then each c_{1i} has a form $c_{1i} = \sum e_j d_{ij}$. Hence we have $\delta(c) = \sum_i (\sum_j e_j d_{ij}) \otimes c_{2i} = \sum_j e_j \otimes (\sum_i (d_{ij} c_{2i})$. Write $x_j = \sum_i (d_{ij} c_{2i})$. Then we see $x_j \in W_0$.

For each $f \in M_2^\vee$, we have $f \cdot c = \sum_j f(e_j)x_j \in \sum_j Rx_j \subseteq W_0$, and hence $M_2^\vee \cdot c \subseteq \sum_j Rx_j$. On the other hand, define $g_i \in M_2^\vee$ by $g_i(e_j) = 1$ when $j = i$ and $= 0$ when $j \neq i$. Then we see $g_i \cdot c = x_i$ and hence $x_i \in M_2^\vee \cdot c$. Thus we have $M_2^\vee \cdot c = \sum_j Rx_j$ and hence $M_2^\vee \cdot cR = \sum_j Rx_j R$.

Now we show that $\delta(M_2^\vee \cdot cR) \subseteq C \otimes (M_2^\vee \cdot cR)$. It suffices to show that $\delta(f \cdot c) = \sum f \cdot e_j \otimes x_j$ since the right hand one is in $C \otimes (M_2^\vee \cdot cR)$.

In fact,

$$\begin{aligned}\delta(f\cdot c) &= \delta\sum_{(c)} f(e_j)x_j\\ &= \sum_{(c)} f(e_j)\sum_{(x_j)} x_{j1}\otimes x_{j2}\\ &= (\zeta(f)\otimes 1_C)\sum e_j\otimes\sum x_{j1}\otimes x_{j2} = (\zeta(f)\otimes 1)(1\otimes\delta)\delta(c)\\ &= (\zeta(f)\otimes 1)(\delta\otimes 1)\delta(c) = (\zeta(f)\otimes 1_C)\sum_{(c)}\sum e_{j1}\otimes e_{j2}\otimes x_j\\ &= \sum_{(c)} f(e_{j1})e_{j2}\otimes x_j = \sum f\cdot e_j\otimes x_j\in C\otimes(M_2^{\vee}\cdot c).\end{aligned}$$

Thus we have shown that $(M_2^{\vee}\cdot cR)$ is a left subcomodule of C containing c. Similarly, we can show that $(Rc\cdot M_2^{*})$ is a right subcomodule of C. □

Thus we have the following:

Proposition 4.3. *Let R be a commutative principal ideal domain, and (C,δ,ε) an R-coalgebra whose underlying module is torsion-free. Then for each $c\in C$ there is a left subcomodule and a right subcomodule of C the underlying modules of which are cauchy and contain c. Moreover, a finite sum of such left (resp. right) subcomodules is again a left (resp. right) subcomodule of C the underlying module of which is cauchy by the local freeness.*

We say that an R-module M satisfies (WLSP) (weak left separation property) if each non-zero $m\in M$ there is $f\in M^*$ such that $f(m)\neq 0$.

For any ring R, it is clear that each left submodule of a left free R-module satisfies (WLSP). If R is a domain, then (WLSP) implies the torsion freeness for any left module since if $f(m)$ is torsion free then so is m.

The author has proved in [14] the following:

Proposition 4.4 (Sun [14]). *For any ring R, we have the following*:

(1) *an R-bimodule M satisfies (WLSP) iff the evaluation morphism $M\to M^{*\vee}$ is an R-bimodule embedding (which can be regarded as a non-commutative analogue of the classical embedding $X\to X^{**}$)*;

(2) *a finitely generated left module M over a Dedekind ring R satisfies (WLSP) iff it is left cauchy*;

(3) *if R is a commutative domain, then (WLSP) is equivalent to torsion freeness for finitely generated modules.*

As a consequence of Proposition 4.4(2), we see that if a module M over a principal ideal domain R satisfies (WLSP), then M is locally free. Thus we have the following, by Proposition 2.1.

Proposition 4.5. *Let R be a principal ideal domain, and (C,δ,ε) an R-coalgebra whose underlying module satisfies (WLSP). Then for each $c\in C$ there is a left subcomodule and a right subcomodule of C the underlying modules of which are cauchy and contain c. Moreover, a finite sum of such left (resp. right) subcomodules is again a left (resp. right) subcomodule of C the underlying module of which is cauchy by the local freeness.*

Remark. The subcomodule $(M_2^\vee \cdot c)$ obviously depends on the choice of representative of $\delta(c)$. So the assignment sending c to $(M_2^\vee \cdot c)$ is not well defined. But we have shown that the sum of any two such subcomodules is a subcomodule which is cauchy. This is enough to meet the need of proving our main result Theorem 5.2 below.

If the underlying module of C is compatibly free, we can simply take $C^\vee$ to replace $M_2^\vee$. In this case, it is independent of the choice of representative of $\delta(c)$, and hence the assignment sending c to $(C^\vee \cdot c)$ is well defined.

Proposition 4.6. *Suppose R is a Dedekind ring and (C,δ,ε) is an R-coalgebra whose underlying module is free. Then for each $c\in C$, $(C^\vee\cdot cR)$ (resp. $(Rc\cdot C^*)$) is a right (resp. left) subcomodule of C which is cauchy.*

The following approach will produce, for a given $c\in C$, a both left and right subcomodule of C which contains c and is cauchy when R is Dedekind. Such a result is interesting since it is closer to the classical fundamental theorem of coalgebras. But it is not necessary for our recovery theorem in this paper.

Lemma 4.7. *For any ring R and any R-coalgebra (C,δ,ε),*

(1) *for any left (resp. right) subcomodule D of C, $D^\perp=\{f\in C^*:\ D\subseteq f^{-1}(0)\}$ is a right (resp. left) ideal of C^*;*

(2) *for any right (resp. left) ideal I of C^*, $I^\perp=\{c\in C:\ f(c)=0 \text{ for all } f\in I\}$ is a left (resp. right) subcomodule of C if the underlying module of C is free;*

(3) *for a both left and right subcomodule D of C, $D^\perp$ is a 2-sided ideal of C^*, and for each 2-sided ideal I of C^*, $I^\perp$ is a both left and right subcomodule of C if the underlying module of C is free.*

Here C^* can be replaced by $C^\vee$.

Proof. (1) For a given left subcomodule D of C, we check that $D^\perp$ is a sub-bimodule of C^*. In fact, for each $r\in R$ and $f\in D^\perp$ and $d\in D$, $(fr)(d)=f(d)r=0$ and $(rf)(d)=f(dr)=0$ since $dr\in D$. So we see that both rf and fr are in $D^\perp$. To show that $D^\perp$ is a right ideal, suppose $f\in D^\perp$ and for any $g\in C^*$ and any $d\in D$, then $(f*g)(d)=g(d_1f(d_2))$ where $\delta(d)=\sum d_1\otimes d_2$ and $d_2\in D$, and hence is equal to zero. Thus $f*g\in D^\perp$.

(2) It is clear that $I^\perp$ is a left submodule. To see that it is also a right submodule, let $c\in I^\perp$, $r\in R$ and $f\in I$, then we have that $rf\in I$ since I is a sub-bimodule of C^*, and hence $(rf)(c)=0$. But we also have $f(cr)=(rf)(c)=0$ so that $cr\in I^\perp$.

To show that $I^{\perp}$ is a left subcomodule of C, assume $m \in I^{\perp}$ and $\delta(m)=\sum_{i=1}^{n} y_i \otimes z_i$ where y_i are basis elements and $\delta(m) \notin C \otimes I^{\perp}$. Then there is $f \in I$ such that $f(z_{i_0}) \neq 0$ for some $\subset_0$. Define $g \in C^*$ such that $g(y_{i_0})=1$ and $g(y_i)=0$ for $i \neq i_0$. Then $fg \in I$ and $0=(fg)(m)=\sum g(y_i f(z_i))=\sum g(f(z_i)y_i)$ since each y_i is central. Hence $0=\sum f(z_i)g(y_i)=f(z_{i_0})g(y_{i_0})=f(z_{i_0})$ – a contradiction. □

Remark. If the ring R satisfies the condition that the pullback P (as shown)

$$\begin{tikzcd} P \arrow[r] \arrow[d] & I^{\perp} \otimes C \arrow[d, "i\otimes 1_C"] \\ C \otimes I^{\perp} \arrow[r, "1_C \otimes i"'] & C \otimes C \end{tikzcd}$$

is equal to $I^{\perp} \otimes I^{\perp}$, then $I^{\perp}$ is a subcoalgebra of C.

Semisimple rings are examples of rings having this property. In fact, in this case, C has a decomposition of $C=I^{\perp} \oplus N$. Thus we see that $(I^{\perp} \otimes C) \cap (C \otimes I^{\perp})=(I^{\perp} \otimes I^{\perp} \oplus I^{\perp} \otimes N) \cap (I^{\perp} \otimes I^{\perp} \oplus N \otimes I^{\perp})=I^{\perp} \otimes I^{\perp}$.

The conclusions in Lemmas 4.7 and 4.6 still hold when we remove the freeness requirement from the coalgebra C but strengthen R to be a finite product of division rings. Notice that in this case we acturally have a subcoalgebra of C.

Lemma 4.8. *Let R be a finite product of division rings Q_i and (C, δ, ε) an R-coalgebra. Then for each 2-sided ideal I of C^*, $I^{\perp}$ is a subcoalgebra of C. Here C^* can be replaced by $C^{\vee}$.*

Proof. It suffices to show $\delta(I^{\perp}) \subseteq (I^{\perp} \otimes C) \cap (C \otimes I^{\perp})$ (it makes sense since C is flat). Since R is semisimple each R-module is a sub-bimodule of a compatibly free module. For each $x \in I^{\perp}$, we have $x=(x_i, x_2, \ldots, x_m)$ where each $x_i \in R$. For simplicity's sake, we say $m=2$ and assume $x_1=(k_1(x_1), k_2(x_1), \ldots, k_n(x_1))$ where $k_i(x_1) \in Q_i$. Again we can assume $n=3$ without loss of generality. Thus we have

$$\begin{aligned} x &= (x_1, x_2) \\ &= ((k_1(x_1), k_2(x_1), k_3(x_1)), k_1(x_2), k_2(x_2), k_3(x_2)) \\ &= ((k_1(x_1), 0, 0), (k_1(x_2), 0, 0)) \quad \text{(denoted by } x^{(1)}) \\ &\quad + ((0, k_2(x_1), 0), (0, k_2(x_2), 0)) \quad \text{(denoted by } x^{(2)}) \\ &\quad + ((0, 0, k_3(x_1)), (0, 0, k_3(x_2))) \quad \text{(denoted by } x^{(3)}). \end{aligned}$$

Since $x \in I^{\perp}$, we see that $x^{(1)}=(1,0,0)x \in I^{\perp}$.

Let $\delta(x^{(1)}) = \sum_{i=1}^{n} y_i \otimes z_i$. Now we use induction on n. When $n=1$, that is, $\delta(x^{(1)}) = y \otimes z$ (we can assume that $z \neq 0$, otherwise $\delta(x) = 0 \in I^{\perp} \otimes C$). Since $\delta(x^{(1)}) = (1,0,0)\delta(x^{(1)})$, we also can assume $y = ((k_1(y),0,0),(k_2(y),0,0))$ and $z = ((k_1(z),0,0),(k_2(z),0,0))$. Since $z \neq 0$ there is a $g \in C^*$ such that $g(z) \neq 0 \in Q_1$ and $g(z) = (q,0,0)$ for some $q \neq 0$. We claim that $y \in I^{\perp}$. If not, then $yg(z) \notin I^{\perp}$ since $y = (1,0,0)y$. So there exists an $f \in I$ such that $f(yg(z)) \neq 0$. Since I is a 2-sided and $f \in I$, we have $g * f \in I$ and hence $0 = (g * f)(x^{(1)}) = \langle \theta_R(g \otimes f), \delta(x^{(1)}) \rangle = \langle \theta_R(g \otimes f), y \otimes z \rangle = f(yg(z)) \neq 0$. Thus we have a contradiction, and so $y \in I^{\perp}$.

Assume that if $\delta(x^{(1)}) = \sum_i^k y_i \otimes z_i$, then $\delta(x^1) \in I^{\perp} \otimes C$ for any $k \leqslant n$. Now let $\delta(x^{(1)}) = \sum_i^{n+1} y_i \otimes z_i$ and $\delta(x^{(1)}) \notin I^{\perp} \otimes C$. Then at least one of the y_i's is not in $I^{\perp}$, say $y_{n+1} \notin I^{\perp}$.

Case 1: $z_{n+1} \in \sum_{i=1}^{n} Rz_i$, i.e., $z_{n+1} = r_1 z_1 + \cdots + r_n z_n$ for some $r_1, \ldots, r_n \in R$. Then $\delta(x^{(1)}) = \sum^n y_i \otimes z_i + y_{n+1} \otimes \sum^n r_i z_i = \sum^n y_i \otimes z_i + \sum^n y_{n+1} r_i \otimes z_i = \sum^n (y_i + y_{n+1} r_i) \otimes z_i$, which is in $I^{\perp} \otimes C$ by induction assumption. Thus we have a contradiction.

Case 2: $z_{n+1} \notin \sum^n Rz_i$. Since C is a direct sum of left ideals and $\sum^n Rz_i$ is a direct summand of C, there is a $g \in C^*$ such that $g(z_{n+1}) = (q,0,0)$ for a non-zero q and $g(z_i) = 0$ for $i \neq n+1$. Then $y_{n+1} g(z_{n+1}) \notin I^{\perp}$ and there is a $f \in I$ such that $f(y_{n+1} g(z_{n+1})) \neq 0$. Again we have $g * f \in I$ and $0 = \langle g * f, x^{(1)} \rangle = \langle \theta_R(g \otimes f), \delta(x^{(1)}) \rangle = \langle \theta_R(g \otimes f), \sum^{n+1} y_i \otimes z_i \rangle = \sum^{n+1} f(y_i g(z_i)) = f(y_{n+1} g(z_{n+1})) \neq 0$. Again we have a contradiction. Thus we have shown that $\delta(x^{(1)}) \in I^{\perp} \otimes C$.

Similarly, we can show that $\delta(x^{(i)}) \in I^{\perp} \otimes C$ for $i = 2,3$, and hence $\delta(x) = \sum_1^3 \delta(x^{(i)}) \in I^{\perp} \otimes C$. To show $\delta(x) \in C \otimes I^{\perp}$, we need the fact that C is right semisimple. $\square$

Lemma 4.9. *Let R be a finite product of division rings Q_i and (C, δ, ε) be an R-coalgebra. Then there is a smallest subcoalgebra of C containing c for each $c \in C$.*

Proof. Let $\{D_i\}$ be all subcoalgebra of C containing c. Write I for the 2-sided ideal $\sum_i D_i^{\perp}$. Then it is clear that $c \in I^{\perp}$. To show that $I^{\perp}$ is the smallest one, it suffices to show that $D = D^{\perp\perp}$ for any subcoalgebra. In fact, if $d \in D^{\perp\perp} \setminus D$, then there is a $g \in C^*$ such that $g(D) = 0$ but $g(d) \neq 0$ since R is semisimple. This is a contradiction. $\square$

Lemma 4.10. *Let N be a sub-bimodule of M. Write $\bar{N} = \{f|N\colon\ f \in M^*\}$ where $f|N$ denotes the restriction of f. Then*

(1) $M^*/N^{\perp} \cong \bar{N} = \{f|N\colon f \in M^*\}$;

(2) *N is isomorphic to a sub-bimodule of $(\bar{N})^{\vee}$ if N satisfies* (WLSP).

Proof. (1) The morphism sending $[f]$ to $f|L$ is a bimodule-isomorphism. (2) follows from Lemma 4.4(2) (for the details, see [14]). $\square$

Proposition 4.11. *Let R be a Dedekind ring and (C, δ, ε) be an R-coalgebra whose underlying module is free. Then for each $c \in C$ there exists a both left and right subcomodule of C which contains c and which is cauchy.*

Proof. Note that C is of course a right C-comodule, which has a right C^*-module action: $c \cdot f = \sum c_1 f(c_2)$ (see Lemma 3.4).

Now consider the right cyclic C^*-module $c \cdot C^*$ denoted by N. We know that N is a finitely generated right R-module (since $c \cdot C^* \subseteq \sum_{(c)} c_{(1)} R$), and that $W = RN$ is cauchy by Lemma 2.5. W is also a right C^*-module since $r(c \cdot f) = r \sum c_1 f(c_2) = \sum r c_1 f(c_2) = (rc) \cdot f$ for each $r \in R$ and each $f \in C^*$. In fact, we knew that W was a right subcomodule of C containing c by Lemma 4.2. But this time we want to produce a both left and right subcomodule of C which contain c and which is cauchy.

Let $I = \{f \in C^*:\ W \cdot f \subseteq \{0\}\}$ (which is equal to $\{f \in C^*:\ N \cdot f \subseteq \{0\}\}$). First we check that I is a sub-bimodule: for each $f \in I$, $r \in R$ and each $n \in N$, we have $(rf)(n) = f(nr) = 0$ and $(fr)(n) = f(n)r = 0$ so that both (rf) and (fr) are in I. Now we check that I is a 2-sided ideal of C^*. It is clear that I is a right ideal of C^*. The fact that I is also a left ideal follows from that fact that $W = (Rc) \cdot C^*$ is a right C^*-module.

Notice that W is in particular left Noetherian. Write $W = \sum_i^n Rx_i$. Define $\pi : C^* \to W^n$ by setting $\pi(f) = (x_1 f, x_2, f, \ldots, x_n f)$. It is well defined since W is a right C^*-module. It is also easily seen that π is right linear and that $ker(\pi) = I$. So we have that $C^*/I \cong im(\pi)$ is isomorphic to a right submodule of W^n and hence is right cauchy.

Since $I \subseteq I^{\perp\perp}$, we have that $C^*/I^{\perp\perp}$ is isomorphic to a sub-bimodule of C^*/I and that $C^*/I^{\perp\perp}$ is right cauchy. But $C^*/I^{\perp\perp}$ is bimodule-isomorphic to $\bar{I}^{\perp}$ by Lemma 4.10(1), we see that $\bar{I}^{\perp}$ is right cauchy, and hence that $(\bar{I}^{\perp})^{\vee}$ is left cauchy (see [14]). Furthermore, $I^{\perp}$ is isomorphic to a submodule of $(\bar{I}^{\perp})^{\vee}$ by Lemma 4.10(2), and hence $I^{\perp}$ is left cauchy. But $I^{\perp}$ is a sub-bimodule of C which is compatibly free, and so $I^{\perp}$ is cauchy by Lemma 2.5. By Lemma 4.7, we see that $I^{\perp}$ is a both left and right subcomodule of C. It remains to prove that $c \in I^{\perp}$ (in fact, $W \subseteq I^{\perp}$). By definition, for each $f \in I$ and each $w \in W$, $w \cdot f = 0$. Thus we have $0 = \langle \varepsilon, w \cdot f \rangle = \varepsilon(\sum w_1 f(w_2)) = \sum \varepsilon(w_1) f(w_2) = f(\varepsilon(w_1) w_2) = f(w)$ by the fact that ε is R-bilinear and f is left linear and by the counitary property. □

In fact the above argument shows the following:

Corollary. *Let R be a Dedekind ring and (C, δ, ε) an R-coalgebra whose underlying module is compatibly free; then for any finite generated submodule N of C, there is a both left and right subcomodule of C which contains N and which is cauchy.*

By Lemma 4.8 and Proposition 4.11, we have the following.

Proposition 4.12. *Let R be a finite product of division rings and (C, δ, ε) an R-coalgebra. Then for any finitely generated submodule N of C, the smallest subcoalgebra containing N exists and is cauchy.*

5. Recovery of coalgebra

We say that a coalgebra (C, δ, ε) satisfies the *functorial fundamental theorem of comodule* (FFTC) if for each $c \in C$, there is a subcomodule D of C such that D contains c and D is cauchy, and if either the D is independent of the choice of the representative of $\delta(c)$ or the set $\mathscr{S}$ of all such subcomodules is a filter or cofilter under inclusion order. This last means that, for each pair of $D_1, D_2 \in \mathscr{S}$, there is a $D \in \mathscr{S}$ such that $D \subseteq D_1 \cap D_2$ or $D \supseteq D_1 + D_2$.

Thus, each of the conditions in Propositions 4.3, 4.5, 4.12 and 4.11 ensures that all the coalgebras there satisfy (FFTC).

To apply Theorem 1.0, we have to show that if an object $M \in (R, R)$-BiMod is right cauchy (in particular, is cauchy) then M has a left dual. However this has been shown by Street in his Notes [12, Theorem 5.2].

Another thing we have to know is that the category (R, R)-BiMod is a closed tensor category (that is, there exist both left and right internal homs for each object). In fact, for any $M \in (R, R)$-BiMod, $M^\vee$ is its left internal hom and M^* is its right internal hom.

Now we are ready to prove the following by mimicking the proof in [12, Proposition 1] or in [11, Lemma 2.2.1].

Proposition 5.1. *Let* (C, δ, ε) *be an* R*-coalgebra which satisfies* (FFTC). *Then* $\mathrm{coend}(U) \cong C$ *as coalgebras where* U *is the forgetful functor* $\mathrm{Comod}(C)_c \to (R, R)$-BiMod.

Proof. We have to show that C has the universal property of $\mathrm{coend}(U)$: that is, that the assignment Ψ sending $f : C \to X$ to $(1 \otimes f)\delta$ determines a bijection between R-module morphisms $f \in \hom_R(C, X)$ and families of R-module morphisms $\theta_M : U(M) \to U(M) \otimes X$ natural in $M \in \mathrm{Comod}(C)_c$. Given a family θ_M, we must define $f(c)$ for each $c \in C$. Let M be any subcomodule of C which contains c and is cauchy. Put $f(c) = (\varepsilon \otimes 1)\theta_M(c)$. This is independent of the choice of M since θ_M is natural in M and C satisfies (FFTC). This gives the inverse to Ψ. $\square$

Theorem 5.2. *Let* (C, δ, ε) *be an* R*-coalgebra. Then* $\mathrm{coend}(U) \cong C$ *as coalgebras, where* U *is the forgetful functor* $\mathrm{Comod}(C)_c \to (R, R)$-BiMod, *if the coalgebra* C *satisfies one of the following properties*:

(1) R *is a commutative principal ideal domain and the underlying module* $U(C)$ *is torsion free*;

(2) R *is a non-commutative principal ideal domain and the underlying module* $U(C)$ *satisfies* (WSLP).

(3) R *is a Dedekind ring, and* $U(C)$ *is free*;

(4) R *is a Dedekind semilocal ring and* $U(C)$ *is projective*;

(5) R *is a finite product of division rings and* C *is an arbitrary* R*-coalgebra.*

The examples of free R-modules above include all polynomial ring $R\langle x_1, x_2, \ldots, x_n\rangle$, (in particular the polynomial ring over the real differential operator ring $B_1(R)$ and all complex Weyl algebra $A_n(\mathscr{C})$); and all group algebras $R(G)$ over a Dedekind R.

Remark. If the ground ring R is commutative, then the constructed coalgebra coend(F) becomes a Hopf algebra by Theorem 1.0.

For a general ring R, the category (R,R)-BiMod is not braided in general. So coend(F) is merely an R-coalgebra.

However, it is possible to find a full subcategory of (R,R)-BiMod which contains all free R-modules and which is symmetric and cocomplete. Thus these coend(F) become Hopf algebras (for the details see the author's [14]).

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